

# Bruhat strata for Shimura varieties of PEL type

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**ABSTRACT.** We study the Bruhat stratification for Shimura varieties of PEL type. In the Siegel case this stratification is a scheme-theoretic variant of the stratification by the  $a$ -number. We show that all Bruhat strata are smooth and determine their dimensions. We also prove that the closure of a Bruhat stratum is a union of Bruhat strata and describe which Bruhat strata are contained in the closure of a given Bruhat stratum.

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## Introduction

### Background

Let  $X$  be an abelian variety over a perfect field  $k$  of characteristic  $p > 0$ . An important invariant of  $X$  is its  $a$ -number  $a(X) := \dim_k \operatorname{Hom}(\alpha_p, X)$ , where  $\alpha_p := \operatorname{Ker}(\operatorname{Frob}: \mathbb{G}_a \rightarrow \mathbb{G}_a)$ . One always has  $0 \leq a(X) \leq \dim(X)$  with  $a(X) = 0$  (resp.  $a(X) = \dim(X)$ ) if and only if  $X$  is an ordinary abelian variety (resp.  $X$  is a superspecial abelian variety). The stratification by  $a$ -number on moduli spaces of polarized abelian varieties has been studied intensively for instance by G. van der Geer in [vdG]. F. Oort has shown that in every non-ordinary Newton stratum of the moduli spaces of principally polarized abelian varieties the stratum where the  $a$ -number is equal to 1 is open and dense in that Newton stratum. He then used this fact to prove a conjecture by Grothendieck which says that the closure of Newton stratum corresponding to a (concave) Newton polygon  $\nu$  is the union of those Newton strata corresponding to the Newton polygons lying below  $\nu$  ([Oo]).

### Main results

In this paper we study the Bruhat stratification introduced in [Wd2] for good reductions of Shimura varieties of PEL type. These good reductions are still moduli spaces of abelian varieties with additional structures. We show in Example 2.4 that for the Siegel case (i.e. for the moduli space of principally polarized abelian varieties in characteristic  $p > 0$ ) the Bruhat stratification is a scheme-theoretic variant of the stratification by the  $a$ -number.

In general, the Bruhat stratification is a decomposition of the special fiber  $\mathcal{A}_0$  of the PEL-moduli space into locally closed subspaces. These subspaces can roughly be described as follows. The first de Rham homology  $H_1^{\text{DR}}(\mathcal{X}/\mathcal{A}_0)$  of the universal abelian scheme  $\mathcal{X}$  over  $\mathcal{A}_0$  with all its additional structures is endowed with two local direct summands, the Hodge filtration and the conjugate filtration (see (2.2) for a precise definition). The Bruhat strata are then the loci, where these two filtrations with all their additional structures are in a fixed relative position. In the case of the moduli space of principally polarized abelian varieties of dimension  $g$ ,  $H_1^{\text{DR}}(\mathcal{X}/\mathcal{A}_0)$  is a locally free module of rank  $2g$  endowed with a symplectic pairing given by the polarization, and the Hodge  $C$  and the conjugate filtration  $D$  are totally isotropic local direct summands of  $H_1^{\text{DR}}(\mathcal{X}/\mathcal{A}_0)$  of rank  $g$ . Then the Bruhat strata are simply the locally closed subspaces  ${}^a\mathcal{A}$ , where  $C + D$  is a local direct summand of  $H_1^{\text{DR}}(\mathcal{X}/\mathcal{A}_0)$  of some fixed rank  $2g - a$ . Moreover  ${}^a\mathcal{A}(\overline{\mathbb{F}}_p)$  consists of those principally polarized abelian varieties  $(X, \lambda)$  over  $\overline{\mathbb{F}}_p$  with  $a(X) = a$ .

In general, the Bruhat stratification is indexed as follows. Let  $\mathbb{G}$  be the reductive group over  $\mathbb{Q}$  of the PEL-Shimura datum. As we assume that the Shimura variety has good reduction at  $p$ , the group  $\mathbb{G}_{\mathbb{Q}_p}$  has a reductive model  $\tilde{G}$  over  $\mathbb{Z}_p$ . Let  $G/\mathbb{F}_p$  its special fiber and denote by  $(W, I)$  its Weyl group together with its set of simply reflections. Let  $[\mu]$  be the minuscule conjugacy class of cocharacters of  $G_{\overline{\mathbb{F}}_p}$  given by the Shimura datum. It determines a conjugacy class of parabolic subgroups of  $G_{\overline{\mathbb{F}}_p}$  and hence a subset  $J \subseteq I$  (see (1.2) for its precise definition). The (geometric) Frobenius acts on  $(W, I)$  via an automorphism  $\bar{\varphi}$  of Coxeter systems. We set  $K := \bar{\varphi}(J)^{\text{opp}}$ , where  $(\ )^{\text{opp}}$  denotes the opposite type. The field of definition of  $J$  is the finite extension  $\kappa$  of  $\mathbb{F}_p$  over which the special fiber  $\mathcal{A}_0$  of the PEL-moduli space is defined. Let  $\Gamma_J$  the Galois group of  $\kappa$ .

We let  ${}^JW^K$  be the set of elements in  $w \in W$  that are of minimal length in their double coset  $W_J w W_K$ , where  $W_J$  and  $W_K$  are the subgroups of  $W$  generated by  $J$  resp.  $K$ . Then the Galois group  $\Gamma_J$  acts on  ${}^JW^K$  and the Bruhat stratification is a decomposition

$$\mathcal{A}_0 = \bigcup_{[x] \in \Gamma_J \backslash {}^JW^K} [x]\mathcal{A}$$

into locally closed subspaces. Our main result is the following (Corollary 4.2):

**Theorem 1.** *The Bruhat strata  $[x]\mathcal{A}$  are smooth of dimension  $\ell(x^{J,K})$  (see (4.1) for the definition of  $x^{J,K}$ ). In particular all Bruhat strata are non-empty. The closure of a Bruhat stratum is given by*

$$\overline{[x]\mathcal{A}} = \bigcup_{[x'] \leq [x]} [x']\mathcal{A},$$

where  $\leq$  denotes the partial order on  $\Gamma_J \backslash {}^J W^K$  induced by the Bruhat order on the Coxeter group  $W$ .

In the Siegel case we show in Example 4.3 that one reobtains the known formulas for the dimension of the  $a$ -number strata (e.g., see [vdG]).

There are two essential tools to prove Theorem 1. The first is the description of the Bruhat stack  $\mathcal{B}_{J,K}$  (2.1) obtained in [Wd2]. The second tool is to show that the morphism

$$a: \mathcal{A}_0 \rightarrow \mathcal{B}_{J,K},$$

which sends an abelian variety  $X$  with additional structure to the triple consisting of  $H_1^{\text{DR}}(X)$  (with its additional structure), the Hodge filtration, and the conjugate filtration, is a smooth morphism (Theorem 4.1).

In [ViWd] Viehmann and the author defined and studied the Ekedahl-Oort stratification for Shimura varieties of PEL type. By construction every Bruhat stratum is a union of Ekedahl-Oort strata. Using the results of [Wd2] we describe in Proposition 3.1 which Ekedahl-Oort strata are contained in a given Bruhat stratum. We deduce the following result (Theorem 3.4):

**Theorem 2.** *There exists a unique open Bruhat stratum  $^{[\tilde{x}]} \mathcal{A}$  which is also dense in  $\mathcal{A}_0$ . Moreover the following assertions are equivalent.*

- (i)  *$^{[\tilde{x}]} \mathcal{A}$  is equal to the generic Newton stratum.*
- (ii) *The field of definition  $\kappa$  of  $J$  is equal to  $\mathbb{F}_p$ .*
- (iii) *There exists an abelian variety  $X$  with additional structure in  $\mathcal{A}_0(\overline{\mathbb{F}}_p)$  which is ordinary.*

## Contents of the paper

In Section 1 we introduce the PEL-moduli spaces and all other notations. We also very briefly recall the Newton stratification. In Section 2 we define the Bruhat stratification of the special fiber of the PEL-Moduli space and show that it generalizes the  $a$ -number stratification in the Siegel case.

We compare the Ekedahl-Oort and the Bruhat stratification and deduce Theorem 2 in Section 3. Section 4 contains Theorem 1 and its proof using a general result about deformations of opposite parabolic subgroups. This result is shown in Section 5.

## 1 Moduli spaces of PEL type with good reduction

Let  $S$  be a scheme over  $\mathbb{F}_p$  and let  $q$  be a power of  $p$ . The pullback of a scheme or a sheaf or a morphism over  $S$  under the  $q^{\text{th}}$  power Frobenius morphism  $S \rightarrow S$  is denoted by  $(\ )^{(q)}$ .

In this section we recall the notion of Shimura-PEL-data and their attached moduli spaces. Our main references are Kottwitz [Kot] and Rapoport and Zink [RaZi].

## The PEL moduli space $\mathcal{A}$

Let  $\mathcal{D} = (\mathbb{B}, *, V, \langle \cdot, \cdot \rangle, O_{\mathbb{B}}, \Lambda, h)$  denote an integral Shimura-PEL-datum that is unramified at a prime  $p > 0$  in the sense of [ViWd] Section 1.1. Let  $\mathbb{G}$  be the associated reductive group over  $\mathbb{Q}$ , and denote by  $[\mu]$  the associated conjugacy class of cocharacters of  $\mathbb{G}$ . We assume that  $\mathbb{G}$  is connected, i.e., we exclude the case (D) of [ViWd] Remark 1.1.

Let  $\bar{\mathbb{Q}}$  be the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ . Then  $[\mu]$  is already defined over  $\bar{\mathbb{Q}}$ . Let  $E$  be the reflex field associated with  $\mathcal{D}$ , i.e. the field of definition of  $[\mu]$ . It is a finite extension of  $\mathbb{Q}$  contained in  $\bar{\mathbb{Q}}$ . Let  $\bar{\mathbb{Q}}_p$  be an algebraic closure and fix an embedding  $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ . This determines a place  $v$  of  $E$  over  $p$ . Let  $E_v \subseteq \bar{\mathbb{Q}}_p$  be the  $v$ -adic completion of  $E$ ,  $O_{E_v}$  its ring of integers, and let  $\kappa$  be its residue field. The assumption on  $\mathcal{D}$  to be unramified implies that  $E_v$  is an unramified extension of  $\mathbb{Q}_p$ . Let  $\bar{\kappa}$  be the residue field of the ring of integers of  $\bar{\mathbb{Q}}_p$ . This is an algebraic closure of  $\kappa$ .

Let  $\mathbb{A}_f^p$  be the ring of finite adeles over  $\mathbb{Q}$  with trivial  $p$ -th component and let  $C^p \subset \mathbb{G}(\mathbb{A}_f^p)$  be a compact open subgroup. We denote by  $\mathcal{A} = \mathcal{A}_{\mathcal{D}, C^p}$  the moduli space defined by Kottwitz [Kot] §5 (see also [ViWd] Section 1).

Then  $\mathcal{A}$  is an algebraic Deligne-Mumford stack which is smooth over  $O_{E_v}$ . If  $C^p$  is sufficiently small,  $\mathcal{A}$  is representable by a smooth quasi-projective scheme over  $O_{E_v}$  (see loc. cit. or [Lan] 1.4.1.11 and 1.4.1.13). We denote its special fiber by

$$\mathcal{A}_0 = \mathcal{A}_{\mathcal{D}, C^p, 0} := \mathcal{A}_{\mathcal{D}, C^p} \otimes_{O_{E_v}} \kappa.$$

## Group theoretical notation

Let  $\tilde{G}$  be the  $\mathbb{Z}_p$ -group scheme of  $O_{\mathbb{B}}$ -linear symplectic similitudes of  $\Lambda$ . This is a reductive group scheme over  $\mathbb{Z}_p$  whose generic fiber is  $\mathbb{G}_{\mathbb{Q}_p}$ . We denote by  $G$  its special fiber. This is a connected reductive group over  $\mathbb{F}_p$  and hence quasi-split (as any reductive group over a finite field). We fix a maximal torus  $T$  of  $G$  and a Borel subgroup  $B$  of  $G$  containing  $T$  (both defined over  $\mathbb{F}_p$ ). For every  $\mathbb{F}_p$ -algebra  $R$  we set

$$G_R := G \otimes_{\mathbb{F}_p} R, \quad T_R := T \otimes_{\mathbb{F}_p} R, \quad B_R := B \otimes_{\mathbb{F}_p} R$$

We denote by  $X^*(T)$  (resp.  $X_*(T)$ ) the group of characters (resp. of cocharacters) of  $T_{\bar{\kappa}}$ . Let  $(W, I)$  be the Weyl group together with its set of simple reflections of  $(G, B, T)$ . We denote by  $w_0 \in W$  the element of maximal length.

The Galois group  $\Gamma := \text{Gal}(\bar{\kappa}/\mathbb{F}_p)$  acts on  $X^*(T)$ , on  $X_*(T)$ , and on the Coxeter system  $(W, I)$ . It is topologically generated by the geometric Frobenius automorphism  $\sigma \in \Gamma$  (i.e.  $\sigma^{-1}$  is the automorphism  $x \mapsto x^p$  of  $\bar{\kappa}$ ). The automorphism of the Coxeter systems  $(W, I)$  induced by  $\sigma$  is denoted by  $\bar{\varphi}$ . Note that  $\bar{\varphi}$  is also the automorphism induced by the geometric Frobenius  $\varphi: G_{\mathbb{F}_p} \rightarrow G_{\mathbb{F}_p}$ .

For any subsets  $J, K \subseteq I$ , we denote by  $W_J$  the subgroup of  $W$  generated by  $J$  and by  ${}^JW$  (resp.  $W^K$ , resp.  ${}^JW^K$ ) the set of  $w \in W$  that are of minimal length in the left coset  $W_J w$  (resp. in the right coset  $w W_K$ , resp. in the double coset  $W_J w W_K$ ). Then  ${}^JW^K = {}^JW \cap W^K$ .

For  $J \subseteq I$  let  $\kappa(J)$  be its field of definition, i.e.  $\kappa(J)$  is the finite extension of  $\mathbb{F}_p$  in  $\bar{\kappa}$  such that

$$(1.1) \quad \Gamma_J := \text{Gal}(\bar{\kappa}/\kappa(J)) = \{ \gamma \in \Gamma ; \gamma(J) = J \}$$

Let  $P_J \subseteq G_{\kappa(J)}$  be the unique parabolic subgroup of type  $J$  containing  $B$ , and let  $\text{Par}_J = G_{\kappa(J)}/P_J$  the projective  $\kappa(J)$ -scheme of parabolics of type  $J$  of  $G$ .

Let  $\tilde{T}$  be a maximal torus of  $\tilde{G}$  such that  $\tilde{T} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = T$  (such a  $\tilde{T}$  always exists because the scheme of maximal tori of  $\tilde{G}$  is smooth, [SGA3] Exp. XV, 8.15) and let  $\mathbf{T}$  be its generic fiber. Let  $\mathbb{Q}_p^{\text{nr}}$  denote the maximal unramified extension of  $\mathbb{Q}_p$  in  $\bar{\mathbb{Q}_p}$  and identify  $\text{Gal}(\mathbb{Q}_p^{\text{nr}}/\mathbb{Q}_p)$  with  $\Gamma$ . Then the cocharacter group  $X_*(\mathbf{T} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\text{nr}})$  is isomorphic to  $X_*(T)$  as  $\Gamma$ -modules. Every element in the conjugacy class  $[\mu]$  is conjugate via some element in  $\mathbf{G}(\mathbb{Q}_p^{\text{nr}})$  to an element of  $X_*(\mathbf{T} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\text{nr}})$  which is unique up to conjugation with an element of the normalizer of  $\mathbf{T} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\text{nr}}$ . Thus via the identification  $X_*(\mathbf{T} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\text{nr}}) = X_*(T)$  we may consider the conjugacy class  $[\mu]$  as an element of  $X_*(T)/W$ .

Let  $\mu$  be the  $B$ -dominant representative in  $X_*(T)$  in the conjugacy class  $[\mu]$ . As  $G$  is quasi-split, it is defined over the field of definition of its conjugacy class, i.e. over  $\kappa$ . We set

$$(1.2) \quad J := \{ i \in I ; \langle \mu, \alpha_i \rangle = 0 \},$$

where  $\alpha_i \in X^*(T)$  is the simple root corresponding to  $i \in I$ . We also set

$$(1.3) \quad K := {}^{w_0}\bar{\varphi}(J).$$

As  $\kappa$  is the field of definition of  $[\mu]$ , one has for the fields of definitions of  $J$  and  $K$

$$\kappa(J) = \kappa(K) = \kappa.$$

## The Newton stratification and the $\mu$ -ordinary locus

As in [ViWd] Section 7.2 we denote by

$$\text{Nt}: \mathcal{A}_0 \rightarrow B(\mathbb{G}_{\mathbb{Q}_p}, \mu)$$

the map that sends each point  $s$  of  $\mathcal{A}_0$  to the isogeny class of the  $p$ -divisible group with  $\mathcal{D}$ -structure given by a geometric point lying over  $s$ . For  $b \in B(\mathbb{G}_{\mathbb{Q}_p}, \mu)$  we denote by  $\mathcal{N}_b := \text{Nt}^{-1}(b)$  the corresponding Newton stratum.

The set  $B(\mathbb{G}_{\mathbb{Q}_p}, \mu)$  is finite and partially ordered. It contains a unique maximal element  $b_\mu$  and the corresponding Newton stratum  $\mathcal{N}_\mu := \mathcal{N}_{b_\mu}$  is called the  $\mu$ -ordinary Newton stratum. It is open and dense in  $\mathcal{A}_0$  by the main result of [Wd1].

## 2 Definition of the Bruhat stratification

In [ViWd] 2.1 Viehmann and the author constructed a morphism  $\zeta$  from  $\mathcal{A}_0$  to the stack of so-called  $\mathcal{D}$ -zips which we identified with the stack of  $G$ -zips of type  $\mu$  (in the sense of [PWZ2]) in [ViWd] 4. The corresponding zip stratification ([PWZ2] (3.27) or [Wd2] Definition 2.8) is nothing but the Ekedahl-Oort stratification. As explained in [Wd2] we also obtain a Bruhat stratification by composing  $\zeta$  with the canonical morphism from the stack of  $G$ -zips of type  $\mu$  into the Bruhat stack

$$(2.1) \quad \mathcal{B}_{J,K} := [G_\kappa \backslash (\text{Par}_J \times \text{Par}_K)]$$

defined in [Wd2]. Here  $[G \backslash X]$  denotes the quotient stack if a group scheme  $G$  acts from the left on a scheme  $X$ . We denote this composition by

$$a: \mathcal{A}_0 \rightarrow \mathcal{B}_{J,K}.$$

We will now give a more direct description of this morphism.

### $\mathcal{D}$ -module structure on the De Rham homology

Let  $S$  be a  $\kappa$ -scheme and let  $(A, \iota, \lambda, \eta)$  be an  $S$ -valued point of  $\mathcal{A}_0$ . Let  $\mathbb{D}(A[p])$  be the *covariant* Dieudonné crystal of the  $p$ -torsion and let  $M(A)$  be its evaluation in the trivial PD-thickening  $(S, S, 0)$ . Then there is an  $\mathcal{O}_S$ -linear functorial isomorphism  $M(A) \cong H_1^{\text{DR}}(A/S)$ , where the De Rham homology is defined as the  $\mathcal{O}_S$ -linear dual of the De Rham cohomology  $H_{\text{DR}}^1(A/S)$  ([BBM] Prop. 2.5.8). Thus  $M(A)$  is a locally free  $\mathcal{O}_S$ -module of rank  $\dim_{\mathbb{Q}}(V)$ . Via functoriality it is endowed with an  $O_{\mathbb{B}}$ -action (induced by  $\iota$ ) and with a similitude class of perfect alternating forms (induced by  $\lambda$ ), i.e.  $M(A)$  is equipped with the structure of a  $\mathcal{D}$ -module in the following sense.

**Definition 2.1.** Let  $T$  be a  $\mathbb{Z}_p$ -scheme. A  $\mathcal{D}$ -module over  $T$  is a locally free  $\mathcal{O}_T$ -module  $M$  of rank  $\dim_{\mathbb{Q}}(V)$  endowed with an  $O_{\mathbb{B}}$ -action and the similitude class of a symplectic form  $\langle \cdot, \cdot \rangle$  such that  $\langle bm, m' \rangle = \langle m, b^* m' \rangle$  for all  $b \in O_{\mathbb{B}}$  and local sections  $m, m'$  of  $M$ .

An *isomorphism*  $M_1 \xrightarrow{\sim} M_2$  of  $\mathcal{D}$ -modules over  $T$  is an  $\mathcal{O}_T \otimes O_{\mathbb{B}}$ -linear symplectic similitude.

Here we call two perfect pairings  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$  on a finite locally free  $\mathcal{O}_T$ -module  $M$  *similar* if there exists an open affine covering  $T = \bigcup_j V_j$  and for all  $j$  a unit  $c_j \in \Gamma(V_j, \mathcal{O}_{V_j}^\times)$  such that  $\langle m, m' \rangle_2 = c_j \langle m, m' \rangle_1$  for all  $m, m' \in \Gamma(V_j, M)$ .

For a morphism  $f: T' \rightarrow T$  of  $\mathbb{Z}_p$ -schemes and a  $\mathcal{D}$ -module  $M$  there is the obvious notion of a pull back  $f^* \widetilde{M} = \widetilde{M_{T'}}$  of  $M$  to a  $\mathcal{D}$ -module on  $T'$ . Altogether we obtain the category  $\mathcal{D}\text{-Mod}$  of  $\mathcal{D}$ -modules fibered over the

category of  $\mathbb{Z}_p$ -schemes. As descent for finite locally free modules is effective for fpqc-morphisms,  $\widetilde{\mathcal{D}\text{-Mod}}$  is a stack for the fpqc topology. In fact it is the classifying stack of  $\tilde{G}$ :

**Proposition 2.2.**  $\widetilde{\mathcal{D}\text{-Mod}} = [\tilde{G} \backslash \text{Spec}(\mathbb{Z}_p)]$ .

Here we endow  $\text{Spec}(\mathbb{Z}_p)$  with the trivial  $\tilde{G}$ -action.

*Proof.* The  $\mathbb{Z}_p$ -module  $\Lambda$  together with the induced  $O_{\mathbb{B}}$ -action and the induced pairing is a  $\mathcal{D}$ -module over  $\mathbb{Z}_p$ . Its automorphism group scheme is  $\tilde{G}$ . Moreover, [RaZi] Theorem 3.16 shows that all  $\mathcal{D}$ -modules are étale locally isomorphic to the  $\mathcal{D}$ -module  $\Lambda_T$ . This shows the claim.  $\square$

We set

$$\mathcal{D}\text{-Mod} := \widetilde{\mathcal{D}\text{-Mod}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = [G \backslash \text{Spec}(\mathbb{F}_p)].$$

We will also use the following torsor  $\mathcal{A}^\#$  over  $\mathcal{A}$ . For every  $O_{E_v}$ -scheme  $S$  the  $S$ -valued points of  $\mathcal{A}^\#$  are given by quintuples  $(A, \iota, \lambda, \eta, \alpha)$ , where  $(A, \iota, \lambda, \eta) \in \mathcal{A}(S)$  and where  $\alpha$  is an  $O_{\mathbb{B}}$ -linear symplectic similitude  $H_1^{\text{DR}}(A/S) \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ .

By Proposition 2.2,  $\mathcal{A}^\#$  is a  $G_{O_{E_v}}$ -torsor over  $\mathcal{A}$  for the étale topology. We also set  $\mathcal{A}_0^\# := \mathcal{A}^\# \otimes_{O_{E_v}} \kappa$ .

### Hodge filtration and conjugate filtration on the De Rham homology

Now let  $(A, \iota, \lambda, \eta, \alpha)$  be an  $S$ -valued point of  $\mathcal{A}_0^\#$ . Then the first de Rham homology  $M(A) = H_1^{\text{DR}}(A/S)$  (i.e., the  $\mathcal{O}_S$ -linear dual of the first hypercohomology of the relative de Rham complex  $\Omega_{A/S}^\bullet$ ) comes with two totally isotropic locally direct summands, namely

$$(2.2) \quad C := \omega_{A^\vee} := f_*^\vee \Omega_{A^\vee/S}^1, \quad D := R^1 f_* (\mathcal{H}^0(\Omega_{A/S}^\bullet))^\perp.$$

Here  $f: A \rightarrow S$  (resp.  $f^\vee: A^\vee \rightarrow S$ ) denotes the structure morphism of  $A$  (resp. of the dual abelian scheme  $A^\vee$ ).

We define  $P$  (resp.  $Q$ ) to be the stabilizer of  $\alpha(C)$  (resp.  $\alpha(D)$ ) in  $G_S$ . Then  $P$  (resp.  $Q$ ) is a parabolic subgroup of type  $J$  (resp.  $K = \bar{\varphi}(J)^{\text{opp}}$ ) of  $G_S$  by [ViWd] Prop. 1.13.

As the formation of  $C$  and  $D$  is functorial, we obtain a morphism

$$a^\# : \mathcal{A}_0^\# \rightarrow \text{Par}_J \times \text{Par}_K.$$

We endow  $\text{Par}_J \times \text{Par}_K$  with the obvious diagonal left action of  $G_\kappa$  and denote by

$$\mathcal{B}_{J,K} := \mathcal{B}_{J,K}(G) := [G_\kappa \backslash (\text{Par}_J \times \text{Par}_K)]$$

the quotient stack. Then  $a^\#$  is  $G_\kappa$ -equivariant and hence it induces a morphism

$$(2.3) \quad a: \mathcal{A}_0 \rightarrow \mathcal{B}_{J,K}.$$

Applying [Wd2] Definition 1.13, the morphism  $a$  yields the *Bruhat stratification of  $\mathcal{A}_0$* , indexed by  $\Gamma_J \backslash {}^J W^K$ , i.e. we obtain a decomposition

$$(2.4) \quad \mathcal{A}_0 = \bigcup_{[x] \in \Gamma_J \backslash {}^J W^K} [x]_{\mathcal{A}}$$

of  $\mathcal{A}_0$  into locally closed substacks.

**Definition 2.3.** The locally closed substacks  $[x]_{\mathcal{A}}$  for  $[x] \in \Gamma_J \backslash {}^K W^J$  are called the *Bruhat strata of  $\mathcal{A}_0$* .

We also define geometric Bruhat strata as follows. Let  $\kappa \subseteq k \subset \bar{\kappa}$  be the splitting field of the reductive group  $G$ . Then  $\text{Gal}(\bar{\kappa}/k)$  acts trivially on  $(W, I)$  and the underlying topological space of  $\mathcal{B}_{J,K} \otimes_\kappa k$  is  ${}^K W^J$ . The residue gerbe  $\mathcal{G}_x$  attached to  $x \in {}^K W^J$  is a locally closed substack of  $\mathcal{B}_{J,K} \otimes_\kappa \kappa(x)$ , where  $\kappa \subseteq \kappa(x) \subseteq k$  is the field of definition of  $x$ . We set

$$(2.5) \quad {}^x \mathcal{A} := a^{-1}(\mathcal{G}_x \otimes_{\kappa(x)} k) \subseteq \mathcal{A}_0 \otimes_\kappa k.$$

This is a locally closed substack of  $\mathcal{A}_0 \otimes_\kappa k$ , called *geometric Bruhat stratum*. Then one has for  $[x] = \Gamma_J \cdot x \in \Gamma_J \backslash {}^J W^K$  a decomposition into open and closed substacks

$$[x]_{\mathcal{A}} \otimes_\kappa k = \coprod_{z \in \Gamma_J \cdot x} {}^z \mathcal{A}.$$

Altogether we obtain a disjoint decomposition into locally closed substacks

$$\mathcal{A}_0 \otimes_\kappa k = \bigcup_{x \in {}^K W^J} {}^x \mathcal{A},$$

which we call the *geometric Bruhat stratification of  $\mathcal{A}_0$* .

**Example 2.4.** Consider the Siegel case, i.e., we choose  $\mathbb{B} = \mathbb{Q}$  in the Shimura-PEL-datum  $\mathcal{D}$ , and hence  $\tilde{G} = \text{GSp}(\Lambda, \langle \cdot, \cdot \rangle)$ . Set  $g := \dim_{\mathbb{Q}}(V)/2$ . Let us explain that in this case the Bruhat stratification is nothing but a scheme-theoretic version of the stratification by the  $a$ -number defined by Oort in [Oo].

The reflex field  $E$  is equal to  $\mathbb{Q}$  and hence  $\kappa = \mathbb{F}_p$ . Then a  $\mathcal{D}$ -module over an  $\mathbb{Z}_p$ -scheme  $S$  is a locally free  $\mathcal{O}_S$ -module of rank  $2g$  endowed with the similitude class of a symplectic pairing. We have  $J = K$  and attaching to a Lagrangian  $C$  in the free  $\mathcal{D}$ -module  $\Lambda_S := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$  (i.e.,  $C$  is a totally isotropic locally direct summand of rank  $g$ ) its stabilizer in  $G_S$  yields a



bijection between the set of Lagrangians in  $\Lambda_S$  and set of parabolic subgroups of type  $J$  of  $G_S$ .

Let  $k$  be a field extension of  $\kappa$ ,  $C_i, D_i \subset \Lambda_k$  Lagrangians ( $i = 1, 2$ ) and let  $P_i = \text{Stab}_{G_k}(C_i)$  and  $Q_i = \text{Stab}_{G_k}(D_i)$  be the corresponding subgroups of type  $J$ . Then two pairs  $(P_1, Q_1)$  and  $(P_2, Q_2)$  are in the same  $G(k)$ -orbit if and only if  $\dim_k(C_1 \cap D_1) = \dim_k(C_2 \cap D_2) \in \{0, \dots, g\}$ . This shows that in this case we can identify  ${}^JW^J$  with  $\{0, \dots, g\}$  via a map  $w \mapsto n(w)$ . Moreover we have  $w \leq w'$  (w.r.t. the Bruhat order) if and only if  $n(w) \geq n(w')$ .

Now assume that  $k$  is perfect and let  $(A, \lambda, \eta, \alpha)$  be a  $k$ -valued point of  $\mathcal{A}_0^\#$ . Denote by  $\sigma$  the absolute Frobenius on  $k$ . Thus  $\alpha$  is a symplectic similitude  $M(A) \cong \Lambda_k := k \otimes_{\mathbb{Z}_p} \Lambda$ , where the symplectic form on  $M(A)$  is induced by  $\lambda$  and on  $\Lambda_k$  by  $\langle \cdot, \cdot \rangle$ . Set  $C := \alpha(\omega_{A^\vee})$  and  $D := \alpha(R^1 f_* (\mathcal{H}^0(\Omega_{A/S}^\bullet))^\perp)$ . These are totally isotropic  $g$ -dimensional subspaces of  $\Lambda_k$ . Recall that for any  $k$ -vector space  $W$  we set  $W^{(p)} = k \otimes_{\sigma, k} W$ . Let  $F: M(A)^{(p)} \rightarrow M(A)$ ,  $V: M(A) \rightarrow M(A)^{(p)}$  be the  $k$ -linear maps induced by Verschiebung and Frobenius, respectively. By transport of structure via  $\alpha$  we obtain  $k$ -linear maps  $F: \Lambda_k^{(p)} \rightarrow \Lambda_k$  and  $V: \Lambda_k \rightarrow \Lambda_k^{(p)}$  such that  $C^{(p)} = V(\Lambda_k)$  and  $D = F(\Lambda_k)$ .

Let  $F^\flat: \Lambda_k \rightarrow \Lambda_k$  be the  $\sigma$ -linear map corresponding to  $F$ , i.e.  $F^\flat(x) = F(1 \otimes x)$  for  $x \in \Lambda_k$ . As  $\sigma$  is bijective, the image of  $F$  and of  $F^\flat$  coincide.

We also denote by  $V^\flat: \Lambda_k \rightarrow \Lambda_k$  the  $\sigma^{-1}$ -linear map corresponding to  $V$ , i.e.,  $V^\flat = \tau \circ V$  with  $\tau: \Lambda_k^{(p)} \rightarrow \Lambda_k$ ,  $(a \otimes x) = \sigma^{-1}(a)x$ . On the other hand, as  $\Lambda_k$  has an  $\mathbb{F}_p$ -rational structure  $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ , there are a canonical  $k$ -linear isomorphism  $\gamma: \Lambda_k^{(p)} \xrightarrow{\sim} \Lambda_k$  and a  $\sigma$ -linear map  $\sigma_\Lambda: \Lambda_k \rightarrow \Lambda_k$ ,  $\alpha \otimes z \mapsto \alpha^p \otimes z$  for  $\alpha \in k$  and  $z \in \Lambda$ . Finally let  $\beta: \Lambda_k \rightarrow \Lambda_k^{(p)}$  the  $\sigma$ -linear map  $x \otimes 1 \otimes x$  for  $x \in \Lambda_k$ . Then

$$(2.6) \quad \sigma_\Lambda \circ \tau = \gamma,$$

$$(2.7) \quad \gamma \circ \beta = \sigma_L.$$

Thus we find

$$\sigma_\Lambda(C) \stackrel{(2.7)}{=} \gamma(\beta(C)) = \gamma(C^{(p)}) = \gamma(V(\Lambda_k)) \stackrel{(2.6)}{=} \sigma_\Lambda(V^\flat(\Lambda_k))$$

and hence  $V^\flat(\Lambda_k) = C$ .

Therefore  $(A, \lambda, \eta)$  is in the Bruhat stratum corresponding to  $w \in {}^JW^J$  if and only if  $\dim_k(F^\flat M(A) \cap V^\flat M(A)) = n(w)$ , i.e. if and only if the point  $(A, \lambda, \eta)$  has  $a$ -number equal to  $n(w)$  in the sense of [Oo].

The morphism  $a$  induces a continuous map  $\mathcal{A}_0(\bar{\kappa}) \rightarrow {}^JW^K$ . This can also be expressed as the following semi-continuity result.

**Remark 2.5.** As usual endow  ${}^JW^K$  with the Bruhat order. For all  $x \in {}^JW^K$  the set

$$\{s \in \mathcal{A}_0(\bar{\kappa}) ; a(s) \leq x\}$$

is closed. Below (Corollary 4.2) we will show that this set is the closure of  $\{s \in \mathcal{A}_0(\bar{k}) ; a(s) = x\}$ .

For instance we can apply this in the Siegel case (Example 2.4). There we identified the partially ordered sets  $({}^JW^K, \leq)$  and  $(\{0, \dots, g\}, \geq)$ . Hence for  $d \in \{0, \dots, g\}$  the locus where the  $a$ -number  $\geq d$  is always a closed subset. Moreover, it is the closure of the locus of points where the  $a$ -number is equal to  $d$ .

### 3 Ekedahl-Oort strata and Bruhat strata

The Ekedahl-Oort stratification defined in [ViWd] is a special case of the zip stratification introduced in [PWZ2]. It is a refinement of the Bruhat stratification and the results of [Wd2] allow to give a simple description of the Ekedahl-Oort strata that are contained in a given Bruhat stratum.

For this recall that the Ekedahl-Oort stratification is a decomposition into locally closed substacks

$$(3.1) \quad \mathcal{A}_0 = \bigcup_{[w] \in \Gamma_J \backslash {}^JW} \mathcal{A}^{[w]}.$$

As above we also have geometric Ekedahl-Oort strata  $\mathcal{A}^w$  for  $w \in {}^JW$  defined over a splitting field  $k$  of  $G$ . By [ViWd] Section 6 and Section 9, the  $\mathcal{A}^w$  are quasi-affine, smooth of dimension  $\ell(w)$ , and the closure of an Ekedahl-Oort stratum is the union of Ekedahl-Oort strata. It suffices to compare geometric Ekedahl-Oort strata and geometric Bruhat strata.

**Proposition 3.1.** *For  $x \in {}^JW^K$  one has*

$${}^x\mathcal{A} = \bigcup_{y \in K \cap x^{-1}JxW_K} \mathcal{A}^{xy}$$

*Proof.* In [Wd2] Proposition 2.10 it is shown that the canonical morphism from the stack of  $G$ -zips of type  $\mu$  to the Bruhat stack  $\mathcal{B}_{J,K}$  is given on the underlying topological spaces by attaching to  $w \in {}^JW$  the unique element of minimal length in  $wW_K$ . This is a surjective map

$$\pi: {}^JW \rightarrow {}^JW^K.$$

Hence for  $x \in {}^JW^K$  one has

$$(3.2) \quad {}^x\mathcal{A} = \bigcup_{w \in \pi^{-1}(x)} \mathcal{A}^w,$$

But by a corollary of a result of Howlett on Coxeter groups (e.g., [PWZ1] Prop. 2.8) one has

$$\pi^{-1}(x) = \{xy ; y \in K \cap x^{-1}JxW_K\}.$$

□

**Corollary 3.2.** *For  $x \in {}^JW^K$  the corresponding Bruhat stratum  ${}^x\mathcal{A}$  consists of a single Ekedahl-Oort stratum if and only if  $K = x^{-1}Jx$ .*

*Proof.* By Proposition 3.1,  ${}^x\mathcal{A}$  consists of a single Ekedahl-Oort stratum if and only if  $K \cap x^{-1}Jx = K$ . As  $J$  and  $K$  have the same number of elements, this condition is equivalent to  $K = x^{-1}Jx$ .  $\square$

**Example 3.3.** Assume that  $J = K$  (e.g., if  $G$  is a group of Dynkin type  $C$ , because then the element  $w_0$  of maximal length is central and  $\bar{\varphi}(J) = J$ ). Let  $1 \in {}^KW^J$  be the element of minimal length. Then  $\pi^{-1}(1) = \{1\}$  and  ${}^1\mathcal{A}$  is the superspecial Ekedahl-Oort stratum (in the sense of [ViWd] Example 4.16).

For  $J \neq K$ ,  ${}^1\mathcal{A}$  is the union of more than one Ekedahl-Oort stratum.

### The maximal Bruhat stratum

Let  $\tilde{x} \in {}^JW^K$  be the element of maximal length and let  $\tilde{x}\mathcal{A}$  be the corresponding Bruhat stratum. It is open and dense in  $\mathcal{A}_0$ : If we consider  ${}^JW^K$  as a topological space, the maximality of  $\tilde{x}$  means that  $\{\tilde{x}\}$  is open in  ${}^JW^K$ . Thus  $\tilde{x}\mathcal{A}$  is an open in  $\mathcal{A}_0$ . Moreover it contains by Proposition 3.1 the  $\mu$ -ordinary Ekedahl-Oort stratum, i.e. the Ekedahl-Oort stratum corresponding to the maximal element  $w_\mu$  in  ${}^JW$  ([ViWd] Example 4.16). By [ViWd] Theorem 6.1 the  $\mu$ -ordinary Ekedahl-Oort stratum is dense in  $\mathcal{A}_0$  and thus  $\tilde{x}\mathcal{A}$  is dense. The density of  $\tilde{x}\mathcal{A}$  will also follow from Corollary 4.2 below.

In general,  $\tilde{x}\mathcal{A}$  is not equal to  $\mu$ -ordinary Ekedahl-Oort stratum (which is also the  $\mu$ -ordinary Newton stratum by the first main result of [Moo]; see also [Wo] for a purely group-theoretical proof).

**Theorem 3.4.** *The following assertions are equivalent.*

- (i) *The generic Bruhat stratum  $\tilde{x}\mathcal{A}$  is equal to the  $\mu$ -ordinary Newton stratum.*
- (ii)  $\bar{\varphi}(J) = J$ .
- (iii)  $E_v = \mathbb{Q}_p$ .
- (iv) *The ordinary locus of  $\mathcal{A}_0$  (i.e. the locus of points  $(A, \iota, \lambda, \eta)$  where the underlying abelian scheme  $A$  is ordinary) is non-empty.*
- (v) *The ordinary locus of  $\mathcal{A}_0$  is equal to the  $\mu$ -ordinary locus.*

*Proof.* Assertion (ii) means that  $\kappa(J) = \mathbb{F}_p$  and thus is equivalent to (iii). The equivalence of (iii), (iv) and (v) is [Wd1] (1.6.3). Thus it remains to show that (ii) is equivalent to the equality of  $\tilde{x}\mathcal{A}$  and the  $\mu$ -ordinary Ekedahl-Oort stratum (by the above mentioned result of Moonen). By Corollary 3.2 this equality holds if and only if  $K = \tilde{x}^{-1}J\tilde{x} = J^{\text{opp}}$ . But by definition  $K = \bar{\varphi}(J)^{\text{opp}}$ . This shows the equivalence of (i) and (ii).  $\square$

**Example 3.5.** The equivalent conditions of Proposition 3.4 are satisfied in the following cases.

- (1) All connected components of the Dynkin diagram of  $G$  are of Dynkin type C.
- (2)  $G$  is split.

## 4 Properties of the Bruhat strata

**Theorem 4.1.** *The morphism  $a$  (2.3) is smooth and surjective.*

Before giving the proof of this theorem we deduce some properties of the geometric Bruhat strata. For this we first introduce the following notation. For  $x \in {}^JW^K$  let  $x^{J,K}$  be the element of maximal length in  ${}^JW \cap W_J x W_K$ . Then  $x^{J,K} = x x_{J,K}$ , where  $x_{J,K}$  is the element of maximal length in  ${}^{K \cap x^{-1}Jx}W_K$  and we have

$$(4.1) \quad \ell(x^{J,K}) = \ell(x) + \ell(x_{J,K})$$

by a result of Howlett ([PWZ1] 2.7 and 2.8).

**Corollary 4.2.** *Let  $x \in {}^JW^K$ .*

- (1) *The corresponding geometric Bruhat stratum  ${}^x\mathcal{A}$  is smooth of pure dimension  $\ell(x^{J,K})$  (4.1). In particular, all Bruhat strata are non-empty.*
- (2) *The closure of  ${}^x\mathcal{A}$  is given by*

$$\overline{{}^x\mathcal{A}} = \bigcup_{x' \leq x} {}^{x'}\mathcal{A},$$

where  $\leq$  denotes the Bruhat order on  ${}^JW^K$ .

*Proof.* As the morphism  $a$  is surjective, all Bruhat strata are non-empty. The  $\mathcal{G}_x \subset \mathcal{B}_{J,K} \otimes \kappa(x)$  be the residue gerbe of  $x$ . Then the cartesian diagram

$$\begin{array}{ccc} {}^x\mathcal{A} & \longrightarrow & \mathcal{A}_0 \otimes \kappa(x) \\ \downarrow x_a & & \downarrow a_{\kappa(x)} \\ \mathcal{G}_x & \longrightarrow & \mathcal{B}_{J,K} \otimes \kappa(x) \end{array}$$

shows that the smoothness of  $a$  implies the smoothness of  $x_a$ . As  $\mathcal{G}_x$  is smooth over  $\kappa(x)$ ,  ${}^x\mathcal{A}$  is smooth over  $\kappa(x)$ .

Finally, as  $a$  is smooth it preserves codimension. By [Wd2] Proposition 1.12,  $\mathcal{G}_x$  has in  $\mathcal{B}_{J,K} \otimes \kappa(x)$  pure codimension  $\dim G - \dim P_J - \ell(x^{J,K})$ . As  $\dim \mathcal{A}_0 = \dim G - \dim P_J$ , we obtain that  ${}^x\mathcal{A}$  is of pure dimension  $\ell(x^{J,K})$ .

As  $a$  is smooth, it is an open morphism. Hence (2) follows from the description of the underlying topological space of  $\mathcal{B}_{J,K}$  ([Wd2] Proposition 1.10) and the fact that for open morphisms taking closures commutes with taking inverse images.  $\square$

*Proof of Theorem 4.1.* The morphism  $a$  is the composition of the morphism  $\zeta$ , which attaches to every point of  $\mathcal{A}_0$  its  $G$ -zip of type  $\mu$ . This morphism is surjective by Theorem 9.1 of [ViWd]. As explained in the proof of Proposition 3.1, the morphism from the stack of  $G$ -zips of type  $\mu$  to the Bruhat stack  $\mathcal{B}_{J,K}$  is also surjective. Therefore  $a$  is surjective.

Moreover, source and target of  $a$  are both smooth over  $\mathrm{Spec} \kappa$ . Therefore it suffices to show that  $a$  is surjective on tangent spaces. By this we mean the following. Let  $k$  be an algebraically closed extension of  $\kappa$ , let  $k[\varepsilon]$  be the ring of dual numbers over  $k$ , and let

$$(4.2) \quad \begin{array}{ccc} \mathrm{Spec} k & \xrightarrow{x} & \mathcal{A}_0 \\ \downarrow & & \downarrow a \\ \mathrm{Spec} k[\varepsilon] & \xrightarrow{\tilde{x}} & \mathcal{B}_{J,K} \end{array}$$

be a 2-commutative diagram. Then we have to show that there exists a morphism  $\mathrm{Spec} k[\varepsilon] \rightarrow \mathcal{A}_0$  which 2-commutes with (4.2).

For this we use Dieudonné displays defined by Zink and Lau ([Zi], [Lau1], [Lau2]; see also [ViWd] 3.1 for a short reminder). We denote the Zink rings of  $k$  and  $k[\varepsilon]$  by  $\mathbb{W}(k)$  and  $\mathbb{W}(k[\varepsilon])$ , respectively. Then  $\mathbb{W}(k)$  is equal to the Witt ring  $W(k)$  and there is a surjective ring homomorphism  $\mathbb{W}(k[\varepsilon]) \rightarrow \mathbb{W}(k)$  whose kernel consists of nilpotent elements. Let  $\mathbb{I}_k = pW(k)$  (resp.  $\mathbb{I}_{k[\varepsilon]}$ ) be the kernel of  $\mathbb{W}(k) \rightarrow k$  (resp. of  $\mathbb{W}(k[\varepsilon]) \rightarrow k[\varepsilon]$ ). We denote the Frobenius on the Zink ring by  $\sigma$  and the pull back via  $\sigma$  by  $(\ )^{(\sigma)}$ .

The  $k$ -valued point  $x$  is given by a tuple  $(A, \lambda, \iota, \eta)$ . Let  $(P, Q, F, F_1)$  be the Dieudonné display attached to its  $p$ -divisible group. It is endowed with a similitude class of perfect alternating forms  $\langle \ , \ \rangle$  induced by  $\lambda$  and an  $O_B$ -action induced by  $\iota$ . In other words,  $P$  is a  $\mathcal{D}$ -module over  $\mathbb{W}(k)$ . As  $W(k)$  is strictly henselian, we may choose by Proposition 2.2 an isomorphism of  $P$  with the  $\mathcal{D}$ -module  $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{W}(k)$ . This defines in particular a  $\mathbb{Z}_p$ -rational structure on the  $\mathcal{D}$ -module  $P$  and an identification  $P/\mathbb{I}_k P = \Lambda_k$ .

The Hodge filtration is given by  $C = Q/\mathbb{I}_k P \subset H_1^{\mathrm{DR}}(A/k) = P/\mathbb{I}_k P$ . Choose a totally isotropic  $O_B$ -invariant direct summand  $C'$  of  $P$  lifting  $C$  and let  $T' \subset P^{(\sigma)} = P$  be an  $O_B$ -invariant totally isotropic complement of  $(C')^{(\sigma)}$ . Then the composition

$$g': P \cong P^{(\sigma)} = (C')^{(\sigma)} \oplus T' \xrightarrow{F_1|_{(C')^{(\sigma)}} \oplus F|_{T'}} P$$

is an automorphism of  $\mathcal{D}$ -modules, i.e.,  $g \in \tilde{G}(\mathbb{W}(k))$ . The image of  $g'(T')$  in  $P/\mathbb{I}_k P$  is the conjugate filtration  $D$ .

The morphism  $\tilde{x}: \mathrm{Spec} k[\varepsilon] \rightarrow \mathcal{B}_{J,K}$  then corresponds to totally isotropic  $O_B$ -invariant direct summands  $\tilde{C}$  and  $\tilde{D}$  of  $\Lambda_{k[\varepsilon]}$  lifting  $C$  and  $D$ , respectively. Applying Lemma 5.1 below to the stabilizers of  $\tilde{D}$  and  $\tilde{C}^{(p)}$ , there exists  $\tilde{g} \in G(k[\varepsilon])$  with  $\tilde{g} \equiv g \bmod \varepsilon$  such that  $\tilde{T} := \tilde{g}^{-1}(\tilde{D})$  is a complement

of  $\tilde{C}^{(p)}$ . As explained in [ViWd] Section 4.1, triples  $(C', T', g')$  as above are parametrized by a smooth scheme and thus we can apply [ViWd] Lemma 3.1 to see that there exists a totally isotropic  $O_B$ -invariant direct summand  $\tilde{C}'$  of  $\Lambda_{\mathbb{W}(k[\varepsilon])}$ , a totally isotropic  $O_B$ -invariant complement  $\tilde{T}'$  of  $(\tilde{C}')^{(\sigma)}$  and  $\tilde{g}' \in \tilde{G}(\mathbb{W}(k[\varepsilon]))$  whose pullback to  $\mathbb{W}(k)$  is  $(C', T', g')$  and whose pullback to  $k[\varepsilon]$  is  $(\tilde{C}, \tilde{T}, \tilde{g})$ . We obtain a Dieudonné display  $(\tilde{P}, \tilde{Q}, \tilde{F}, \tilde{F}_1)$  as follows. We define  $\tilde{P} := \Lambda_{\mathbb{W}(k[\varepsilon])}$  and denote by  $\tilde{Q}$  the inverse image of  $\tilde{C}$  under  $\tilde{P} \rightarrow \tilde{P}/\mathbb{I}_{k[\varepsilon]}\tilde{P} = \Lambda_{k[\varepsilon]}$ . Finally let  $\tilde{F}: \tilde{P}^{(\sigma)} \rightarrow \tilde{P}$  and  $\tilde{F}_1: \tilde{Q}^{(\sigma)} \rightarrow \tilde{P}$  be the unique  $\mathbb{W}(k[\varepsilon])$ -linear maps making  $(\tilde{P}, \tilde{Q}, \tilde{F}, \tilde{F}_1)$  into a Dieudonné display such that the direct sum of the restriction of  $\tilde{F}_1$  to  $(\tilde{C}')^{(\sigma)}$  and the restriction of  $\tilde{F}$  to  $\tilde{T}'$  is given by  $\tilde{g}$ . Then  $(\tilde{P}, \tilde{Q}, \tilde{F}, \tilde{F}_1)$  is a Dieudonné display with symplectic form and  $O_B$ -action lifting  $(P, Q, F, F_1)$  and thus defining a morphism  $\mathrm{Spec} k[\varepsilon] \rightarrow \mathcal{A}_0$  which 2-commutes with (4.2).  $\square$

**Example 4.3.** We consider again the Siegel case, i.e.,  $\mathbb{B} = \mathbb{Q}$  and therefore  $\tilde{G} = \mathrm{GSp}(\Lambda, \langle \cdot, \cdot \rangle)$ . Set  $g := \dim_{\mathbb{Q}}(V)/2$ . We have seen in Example 2.4 that the Bruhat stratification on  $\mathcal{A}(\bar{\kappa})$  can also be described by

$$\mathcal{A}(\bar{\kappa}) = \bigcup_{0 \leq i \leq m} \mathcal{A}_i,$$

where  $\mathcal{A}_i$  denotes the locally closed subvariety of  $\mathcal{A}(\bar{\kappa})$  consisting of principally polarized abelian varieties whose  $a$ -number is equal to  $i$ . By Corollary 4.2 we have

$$\overline{\mathcal{A}}_i = \bigcup_{j \geq i} \mathcal{A}_j$$

and  $\mathcal{A}_i$  is smooth and equi-dimensional of dimension  $d(i)$  with

$$d(i) := \sum_{j=i+1}^g j = \frac{g(g+1) - i(i+1)}{2}.$$

This explicit description of  $\ell(x^{J,J})$  for  $x \in {}^J W^J \cong \{0, \dots, g\}$  follows from the explicit calculation of the length in [ViWd] Appendix A.7, in particular the formulas (12.3) and (12.4) there. Note that for the identification of  ${}^J W^J$  and  $\{0, \dots, g\}$  made here in Example 2.4, the map in loc. cit. (12.4) should be given by  $(\varepsilon_i)_i \mapsto \#\{i; \varepsilon_i = 0\}$ .

## 5 Deforming parabolics in opposite position

Here we prove a group-theoretical lemma used in the proof of Theorem 4.1. Let  $k$  be any field, and let  $G$  be a reductive group over  $k$ . Let  $(W, I)$  be the Weyl group of  $G$  together with its set of simple reflections. Recall that for any  $k$ -scheme  $S$ , two parabolic subgroups  $P$  and  $Q$  of  $G_S$  are called *opposite*

if  $P \cap Q$  is a Levi subgroup of  $P$  and of  $Q$ . We indicate this by writing  $P \bowtie Q$ . Moreover for every Levi subgroup  $L$  of a parabolic subgroup  $P$  there exists a unique parabolic subgroup  $Q$  of  $G_S$  such that  $P \cap Q = L$  and then  $P \bowtie Q$  ([SGA3] Exp.XXVI Théorème 4.3.2). This implies that if  $J \subseteq I$  is the type of  $P$ , the type  $J^{\text{opp}}$  of a parabolic subgroup  $Q$  opposite to  $P$  depends only on  $J$ .

**Lemma 5.1.** *Let  $Z_J$  be the  $k$ -scheme whose  $S$ -valued points are the set of triples  $(P, Q, g)$ , where  $P$  is a parabolic subgroup of  $G_S$  of type  $J$ ,  $Q$  is a parabolic subgroup of  $G_S$  of type  $J^{\text{opp}}$ , and  $g \in G(S)$  such that  ${}^gQ \bowtie P$ . Then the canonical morphism*

$$Z_J \rightarrow \text{Par}_J \times \text{Par}_{J^{\text{opp}}}, \quad (P, Q, g) \mapsto (P, Q)$$

*is smooth.*

*Proof.* We set  $X := \text{Par}_J \times \text{Par}_{J^{\text{opp}}}$ . Let  $\mathcal{P}_J \subset G_{\text{Par}_J}$  be the universal parabolic subgroup of  $G_{\text{Par}_J}$  of type  $J$  and let  $\mathcal{P}'_J$  be its pull back to  $X$  under the first projection. This is a parabolic subgroup of  $G_X$ . Define similarly a parabolic subgroup  $\mathcal{P}'_{J^{\text{opp}}}$  of  $G_X$ . Then the  $X$ -scheme  $Z_J$  is isomorphic to the subscheme of  $G_X$  whose  $S$ -valued points are given by  $\{g \in G_X(S) ; {}^g\mathcal{P}'_{J^{\text{opp}}} \bowtie \mathcal{P}'_J\}$ . This is an open subscheme of  $G_X$  by [SGA3] Exp.XXVI Théorème 4.3.2. As  $G_X$  is smooth over  $X$ , this implies that  $Z_J$  is smooth over  $X$ .  $\square$

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